II.—WHAT DO WE MEAN BY THE QUESTION: IS OUR SPACE EUCLIDEAN?

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EVER since the existence of Non-Euclidean geometry has been widely enough known to reach even philosophic circles philosophers have asked whether our space is Euclidean, and whether there is any means of finding out if it be not Euclidean. That ordinary philosophers should have disagreed in their answers to such questions as these is not surprising; they lacked the mathematical training needed for an intelligent discussion of the subject and many of them were sadly led astray by a popular article by Helmholtz. But it is more surprising that men like the late M. Poincaré and the living Mr. Bertrand Russell should come to quite opposite conclusions on these questions. Both are absolutely competent to appreciate all mathematical points involved; the former was a great mathematician and a respectable philosopher, the latter is an extremely competent mathematician and an eminent philosopher. We must suspect then that their different answers are due to some ambiguity in the question. In this paper I am going to try and clear up some of these ambiguities; and, when this has been done, we may find a probable answer to the above questions.

Obviously the first point to clear up is what is meant by the phrase 'our space'. Until we know precisely what we mean by 'our space' it is useless to ask any further questions about it. The phrase is a peculiar one; it seems to suggest that we have a space of our own as we have a latchkey or as Trinity College has a hall. And this suggests that there may be other spaces owned by other people, just as other people have latchkeys and other colleges have halls. Now in a certain sense it is true that each of us has a private space peculiar to himself and as unique as his latchkey. Such spaces may be called perceptual spaces. We must inquire first what is meant by a private perceptual space.

To answer this question we need to go a little further back and ask what is meant by *a* space. There is of course an ambiguity in this question. In one sense *a* space is any closed piece of extension. In this sense we should call the Great Court of Trinity *a* space. But this is not what is meant in the present question. What is meant is this. Mathematicians talk of Euclidean, and hyperbolic spaces, and now we are beginning to talk about private perceptual spaces ; here we are using space as a general term of which there are different particular instances, just as we use nation as a general term and then distinguish between the English, the French, and the German nations. So the question is : What must all kinds of spaces have in common in order that the common name space may be appropriate to them? I think when we talk of a space we assume the following things. We assume a class of entities which we call points and we assume certain kinds of relations between them and other relations which only relate certain selections of them. Thus a straight line in any space is a certain selection of the points in that space which are related to each other in a certain way and are not related to other points of the space in this way. Similar remarks apply to planes and to other curves and surfaces in the space. The relations and the important kinds of subclasses of related points in the space are or should be named and defined in the definitions of the geometry in question. The axioms will tell you the relations that must be assumed between these primary relations and subclasses. Thus the axiom that two straight lines cannot inclose a space in Euclidean geometry tells you that in that geometry any two subclasses of related points which agree with the Euclidean definition of a straight line will either have one point or none in common. But at present it is important for us to notice that there is something further which does *not* explicitly appear in the definitions, axioms, and postulates of any system of geometry. This is the fact that we certainly draw a distinction between space and matter which is in space. Of course points are unextended, whilst all bits of matter are supposed to have some extension though it may often be an imperceptible one. Thus any bit of matter corresponds to an infinite number of points of the space in which it is supposed to be. But this is not the most important distinction. Matter is supposed to be capable of moving about, and, if we talk of a piece of matter as being in a certain kind of space we say that it moves about in this space. But we cannot say that points of space move about. What precisely does this distinction come to? It comes to this. If we mean to distinguish space from matter we must suppose that pieces of matter are related in a certain peculiar way to points of space. The relation itself is not a simple one. It involves space and matter and time. We say that a piece of matter is at a certain place at a certain time. This means that if we imagine the matter divided up into material unextended points each of them will be at one definite point of space at a certain moment of time. When we say that the piece of matter moves we mean that at a second moment some of its material points are at different spatial points from the ones at which they were at the first moment. We must for completeness distinguish between movements of translation, movements of rotation, dilatations and deformations. It is not important for our present purpose to go into the question of how these are distinguished. Now suppose there be two pieces of matter in the space in question. On the view that there is such a thing as space, that it is distinct from matter, and that matter is in space, we must next carefully distinguish two different kinds of spatial relations which are called by the same name and are liable to be confused. The first kind are the relations between bits of matter; as when we say that Cambridge is sixty miles N.E. of London. The second is the relation between points of space as when we say that the place where Cambridge is is sixty miles N.E. of the place where London is. But this distinction is only a rough first approximation to the distinctions that we must finally make if we are to be in earnest with our view that matter is in space. You obviously cannot talk strictly of *the* distances between Cambridge and London because various parts of Cambridge are at different distances and in different positions relative to various parts of London. Thus the distinction that we have ultimately to recognise is that between the relation of a definite material point in the mass of matter which we call Cambridge to some other definite material point in the mass of matter which we call London and the mutual relation of the geometrical points *at* which these material points are situated. Now to a person who is in earnest with the notion of space there is an important difference between the two kinds of relations. The spatial relation between two material points is not a simple or ultimate thing. It is compounded of the relation between the two geometrical points at which the material points are situated and the relation which each of the material points has to the geometrical point at a given moment. The statement 'the material point A is twelve miles to the S.W. of the material point B at the moment t' means 'the material point A is at the geometrical point α at t and the material point B is at the geometrical point β at t and the geometrical point α is eternally twelve miles S.W. of the geometrical point β.'

This shows us an important distinction between the spatial relations of material points and the similarly named relations between spatial points. The relations between material points may alter with time, those between spatial points are essentially timeless. You can say that A was twelve miles S.W. of B at t_1 and two miles N.E. of it at t_2, but this does not imply that any change has taken place in the relations of the geometrical points α and β. It only means that A or B or both of them have ceased to be at the geometrical points α and β and have come to be at other points α^1 and β^1 which have and always have had the relation that α^1 is two miles N.E. of β^1. It is then of the very essence of the notion of space as distinct from matter that points of space and their geometrical relations are timeless, and that the spatial relations of material points can alter in time owing to the fact that two material points can at different times be at different geometrical points without making any difference to the spatial relation of these geometrical points to each other. If this distinction be forgotten it is impossible to make any clear separation of matter and space. This is of great importance because it cuts out at once certain suggestions of extremely empiricist mathematicians like Clifford, that the space-constant of our space may vary with time and that this may explain certain physical phenomena. Any one who takes such a view as this may be invited to tell us how he distinguishes space from matter and whether he is really doing more than ascribing certain variable qualities to some pervasive medium like the ether.

A further question now arises. Granted that it is a part of what we mean by space as distinct from matter that it shall not vary with time is it also a part of its meaning that it shall be homogeneous? Is it compatible with the notion of space that its measure of curvature though independent of time may vary from place to place? I think not. It is incompatible with our notion of space that absolute position in space should be relevant in any physical law. If, for instance, bodies always changed their size and shape in certain definite ways as they were moved about we should feel it inappropriate to say that this was due merely to the change of their position in space. We should always assume physical causes for these physical changes. And I may remark, we could never be proved to be wrong in doing this. When a finite body alters its position in absolute space this is never the *only* fact about it that alters. It *ipso facto* changes its relative positions to other bodies; hence any change in the body can always be referred to these changes of *relative* position, and this be given physical causes. A similar remark applies to absolute time. It is incompatible with the distinction between time and events in time that any causal law should contain absolute time; for this would mean that one antecedent in a causal law was not an event but a moment and so time would act on matter. But there can never arise the least necessity to employ causal laws which contain absolute time, for the following reason. The only ground that could make any one wish to do this would be if it were established that the universe had been in the same total state at two absolute moments t_1 and t_2, and yet that its history between t_1 and t_2 differed from its history after t_2. They would then be inclined to say: There is no difference in the antecedents of states that follow t_1 and states that follow t_2 on this hypothesis and yet these states differ. Hence they cannot be determined solely by antecedent *events*, but the absolute time at which a state happens must be relevant. But this is quite a mistake. It rests on supposing that, when the state of the universe at any *one* moment is given it ought to be determinate at any other moment; and this view itself rests on the false belief that the series of events is not continuous but that there are next moments. The fact is that the state of the universe at any moment may be a function of its states at several other moments. In that case the history of the world between t_1 and t_2 will be a function, not merely of its state at t_1, but of this and some of its states before t_1. The history of the world after t_2 will be a function of its state at t_2 and of some of its states before t_2. The selection of states before t_1 needed to determine the history of the universe between t_1 and t_2, need not be the same as the selection of states before t_2 needed to determine its history after t_2. Hence the mere fact that the states at t_1 and t_2 are identical is no reason why the states between t_1 and t_2 should be identical with the corresponding ones after t_2. Thus, if these corresponding states are not found to be identical, we still have no reason to suppose that they cannot be determined wholly by antecedent *states*, and therefore no reason to think that they involve any reference to absolute moments of time.

We thus reach the important conclusion that, if we mean to be in earnest with our distinction between space and matter on the one hand and time and events on the other, we must lay down the following conditions for space and time. Space and time are not themselves in time, geometrical and temporal relations are eternal. The only thing that changes in motion is the relation of material points to geometrical points. Again space and time cannot be conceived as capable of causal action on matter. All laws about the changes of matter must simply contain the states of matter at one or more times, *relative* positions, and *differences* of time. They must not contain absolute positions or absolute moments. Space and time must therefore be conceived as homogeneous; they must not have different qualities at certain points or moments from what they have at other points or moments. And we have seen that experience can never force us to any other conclusion. Of course nothing that I have said lends support to the view of certain French philosophers who hold that we can decide at once in favour of Euclidean geometry because it allows of similar figures whilst hyperbolic and elliptic geometry do not. There is nothing incompatible with what I have said about the necessary qualties of space and time in the non-existence of similar figures. That there should be a certain constant relation between lengths of side and magnitudes of angles involves no causal action of space and time on matter, any more than in Euclidean geometry the fact that you could not make a triangle, whose angles were less than two right angles means that Euclidean space acts causally on matter. This relation is eternal and purely geometrical.

The reader will naturally have been wondering during this discussion what right I have to lay down in this confident way the properties that must and those that cannot belong to space and time. He will ask: Are they axioms and self-evident; or are they merely a question of definition? If they be merely a question of definition why should our definition be any better than one which Clifford might have made up for space which should allow space to have a measure of curvature variable with time? The answer to this question will help us a good deal towards a solution of our main problem.

The point to notice is that the distinction between space and matter is not something that we find, but something that we make—to use a somewhat unguarded expression which I will modify later. When I say that we do not find the distinction I mean two things. (1) We clearly do not directly perceive space and directly perceive matter and then compare them and find what are the characteristic qualities of each. It is admitted on all hands that empty space is not perceptible at all, and it must further be admitted that if by matter you mean what physicists mean by matter *it* too is not directly perceptible. But (2) this is not all. Space and matter are not two definite things which are given together in experience and then separated out by analysis. When we hear a musical note we hear a complex unity; subsequent reflexion and comparison enable us to assert that in all notes will be found the two characteristics of pitch and loudness. I should then say that we analysed the two qualities of pitch and loudness out of a complex experienced object and can go on safely to describe the peculiar nature of these two qualities. But this is not the kind of procedure by which we reach the distinction between space and matter and become acquainted with the characteristic peculiarities of each. When we state that the proper interpretation of any relative motion is that of two pieces of matter one at least has come to occupy a new position in a changeless space it cannot be said that we are *merely* analysing a complex given in experience and finding what was in it all along. The whole scaffolding of a space of points in eternal geometrical relations to each other seems quite obviously not to be an element given to us confusedly in an experienced complex and clearly recognised by subsequent reflexion, but to be something added by us to the experienced facts. Of course it is not a case of *mere* addition. We do analyse what we experience into bodies and their relative positions and geometrical relations; but then we treat these bodies as complexes of material points correlated from time to time with various geometrical points whose mutual relations are eternal. The first part of the process is genuine analysis quite comparable to the discovery that a sound has both pitch and loudness; but the second part, the part that introduces space, is quite clearly not just a further analysis of the same kind. It is not a finding of what when found we recognise to have been there all the time; it is an addition made by us involving a special interpretation of the experienced facts.

If now you ask: Why should I accept precisely your definition between space and matter; is it an axiom or a definition or what? we are prepared to answer: Certainly it is not an axiom, but a matter of definition. But it is not arbitrary. The distinction of space and matter, the view that matter moves about in space whilst space remains eternally unchangeable, and the view that time and space do not act on matter:—these are the characteristics of a certain way of describing the experienced facts. It is not the only possible way, but it is the way which common-sense and science have taken. So long as you talk about space at all you presuppose that this method of describing the facts has been adopted, and so long as you do this you must ascribe to space the qualities that I mentioned. To put it in another way. To talk of space and its qualities presupposes that space is something distinct from matter; hence it is useless to try and give space qualities which made it indistinguishable from matter. And a careful consideration of what we really do mean by space will show that we mean something that has the qualities and the relations to matter which I have described.

I must now try and state much more accurately what I meant when I said that space was something that *we* added to our experience and not something found by us. This sounds very subjective and Kantian, and I must make a number of distinctions to avoid misunderstanding. 1. [1] I do *not* think that the shapes, sizes, motions, and spatial relations of perceived objects can in any sense be supplied by our minds; they are found and not made by us just as are colours, sounds, etc. 2. I do mean that, when we interpret relative motion in terms of absolute motion in space, the space and its qualities are neither (*a*) directly experienced like colours; nor (*b*) recognised to be present in what we experience by subsequent reflexion, as pitch and loudness are recognised to have been present in any sound; nor (*c*) reached by inference from what we do experience, as we reach the belief that there are light-waves from our experiences of colours. I will elaborate this last point a little further. It follows from the definition of absolute time and space that our reasons for believing in them can never be an inference from what we perceive such as we use in physics to support our beliefs in imperceptible objects like atoms and ether-waves. All these inferences in physics depend on the view that the inferred entities cause something in what we experience. Now it is part of the definition of space and time that they cause nothing; hence our belief in them cannot be supported by inference from what we directly experience. 3. It seems then that the interpretation of perceived spatial relations and perceived motions in terms of space must be something that we ourselves add to the facts. By this I do not mean that there may not be such a thing as a real geometrical space of

[1] I do not wish to deny that the *sensuous* peculiarity which distinguishes a felt corner from a seen corner may be mind-dependent. But it is no more so than any other sense-datum.

points in eternal relations, but that we have no reason for believing that there is such a space. It is one possible interpretation of observable facts, but it is rather a bizarre interpretation the elements of which are supplied by ourselves. How much of the interpretation is supplied by ourselves and what precisely this means are questions which I defer for the moment. At present we know enough about what is meant by space to be able to return with profit to the question what is meant by *our* space. The further working out of the answer to this problem will do much to clear up this deferred question.

We said that we could talk of private spaces and that these were found in perception. Psychologists talk of percĕptual space and conceptual space, and contrast the two. But the distinction is not quite a happy one. To talk of a perceptual space suggests that it is a kind of space that can be perceived. But no kind of space can be perceived by the senses. What is true is that several of our senses, *e.g.*, those of touch and sight, make us aware of extended wholes in which we can distinguish parts in spatial relations to each other and in relative motion. For example, in sight we become aware of a total field of vision and can see in it visual objects in spatial relations to each other. Again by touch we become aware of tactual objects with various shapes and spatial relations and we may feel these moving about. But the object of sight and the object of touch are not themselves spaces. What would really be meant by a private visual space would be this. Suppose a man were to deal with all his visual experiences on the plan of constructing visual bodies and the space in which they move, the space having those characteristics which we have laid down for all spaces; then the space so constructed would be a private visual space. We must assume that the man takes no heed of any information that he gains from any other sense; the space is to be constructed so as to deal simply with the data of all his sight experiences on the general plan of distinguishing space from things in space. Similarly a private tactual space would be reached by a man who should deal with all the data given him by touch without reference to any other sense, according to the general plan of distinguishing space from matter which moves about in it. Thus to each private sense-space will correspond a special kind of thing: to sight-space will correspond visible things, to touch-space felt things.

We thus see that the distinction between perceptual and conceptual space is not a happy one. All spaces are conceptual in the sense that they are constructed in order to deal with certain sets of experienced objects according to a certain definite plan; all spaces are perceptual in the sense that they are constructed to deal with the extended data of certain senses. But there is an important practical distinction between the private space of any sense in any person and what is commonly called conceptual space. The latter is constructed to deal consistently, according to the fundamental plan of distinguishing space and things in it, with the data of all senses in all people. But the private spaces of the special senses in particular people never have been constructed and perhaps could not be constructed at all. To construct such a space we should have to be sure, *e.g.*, that all the visible objects that we perceive with their visible movements can consistently be regarded as things moving about in a special space with the qualities that we have laid down for all spaces. Whether this could be done at all successfully is by no means certain; there is no *a priori* necessity why the data of each of our senses in abstraction from those of the others should be capable of being dealt with according to this plan; and it is quite certain that it has never been done by any one. We must therefore regard private perceptual spaces as at best constructions which *may* (but may *not*) be possible, and we can say with some confidence that, *if* they be possible, they are most unlikely to be Euclidean.

We are now very near the answer to our question: What do we mean by *our* space for the purpose in hand? We do not mean the private perceptual spaces of any one, for we do not even know whether such spaces be possible. We must mean a space so constructed as to enable us to deal with the data of all senses of all men. This is a rough general way of putting our answer; we must now try to refine it. Let us call this space physical space. The first thing to notice is that, though physical space is defined as a space constructed to enable us *to deal with* the data of all the senses of all men, yet it is not true that the data of any sense of any person are *in* physical space. If the data of any sense of any person be in space at all they are in the private space of that person appropriate to that sense. If there be no such things as private perceptual spaces then no one's sense-data are in any space at all, though of course they have spatial qualities, *i.e.* they are extended, move, and have spatial relations to each other. This apparently startling result arises from the close correlation of space and bodies; every special kind of space involves a special kind of bodies, for space and bodies are the two correlative elements involved in a certain definite way of dealing with an extended whole whose parts have relative motions. The bodies that correspond to physical space are physical bodies; these are not identical with what is perceived by any of our senses; what I see and what you see at the same moment when we say that we are looking at the same body will have slightly different shapes; the body that is supposed to be moving in physical space is not identical with what I see or with what you see, though it is correlated with the sense-data of both of us. The distinction between physical bodies moving in physical space and perceptible bodies moving in some private perceptual space if in a space at all is best seen in the case of dreams. In dreams we see things in various spatial arrangements moving about in various ways; but we never suppose that they are in the same space as chairs and atoms; if they move in a space at all it is in a dream-space appropriate to them. The reason why it is obvious that dream-objects are not in physical space and less obvious that sensible objects are not in physical space is that physical space has been constructed to deal with all or nearly all the objects perceived in waking life but not to deal with dream objects. It follows that there is a close correlation between physical objects and the objects of our senses in waking life, but that there is practically none between dream-objects and physical objects. We therefore have no temptation to think that the objects of dream perception are in physical space. We do not believe this even of the objects of certain sense-perceptions; when we feel a huge hole in a tooth with the tip of our tongue we do not believe that a huge hole exists in any object in physical space. This is because physical space and physical bodies are only constructed to deal with certain important data of sight and touch and not with all perceptual data even of waking life. It is by no means certain that, if we tried to construct a space that should consistently deal with both the data of waking life and those of dreams or even with *all* the data of waking life, we could do it. Our total mass of perceptual data may very well not be susceptible of treatment according to the general plan which distinguishes matter and space.

I think there can now be little doubt as to what we mean by *our* space when we ask whether it is Euclidean. We mean the space of physics which has been constructed to deal consistently with most of the data of waking sight and touch in most people on the general plan of distinguishing space and matter and attributing to space the qualities that I have laid down. Now that we know what is meant by *our* space we can go on to discuss what is meant by asking whether it is Euclidean. Three points arise for discussion. (1) Is this a sensible question? (2) If it has a meaning, is it capable of a single definite answer? and (3) If it be capable of a single definite answer in theory is there any practical way of finding the answer?

To answer these questions we must go more thoroughly than we have yet done into what is meant by saying that *we* construct space. This sounds very subjective. I have already pointed out that I do not mean by the phrase anything like the Kantian notion that the mind makes extension or spatial relations. It finds these just as it finds colours, sounds, etc. But I have only further described what I mean by construction in negative terms. I have argued that to say that we construct space involves at least the two negative propositions, (1) that we do not discover it by analysis of what we find given to the senses, and (2) that we do not infer it as something causally connected with what we perceive. I will now add a more positive determination of this much-abused word 'construction'. The best way to approach this subject is by considering the use of parameters in elementary mathematics. Thus in dealing with ellipses it is customary to introduce a certain angle such that $x = a \cos \phi$ and $y = b \sin \phi$. A closer analogy to construction in the sense in which I use the word is found in the introduction of the parameter θ in dealing with cycloids. We find it simpler to deal with cycloids on the assumption that they have all been generated by circles rolling on straight lines. The parameter θ represents the angle turned through by the circle from the beginning of its motion. Now it is quite clear (1) that not all cycloids really have been made by actual circles rolling on actual straight lines. The arches of Westminster Bridge are arcs of cycloids but they certainly were not made in this way. (2) It is also clear that it is quite irrelevant for all mathematical purposes whether any particular cycloid was actually produced in this way; we can always introduce the parameter θ and deal with cycloids by means of it no matter what is the physical history of any cycloid. Now what I want to suggest is that physical space and physical matter are, so far as we know, just parameters which are introduced when we deal with sensible experience according to a certain plan. Why we insist on dealing with sensible experience according to this plan is not obvious. If anything at all be left of Kant's intuitional theory of space I think it may reduce to this that we have an innate tendency to deal with sensible experience according to the plan of separating space and matter, ascribing to space the general qualities that I have laid down, and conceiving matter as moving about in it. But I do not wish to insist on this suggestion. The more important point for us to notice is that actual experience has proved that the steady pursuit of this general plan of dealing with the sensible world has had very great success; we have by its means made this world and its changes more and more intelligible and predictable. But this does not prove that it is the only possible plan, or that there may not be a yet more successful alternative. Many alternative plans are suggested, *e.g.* by Dr. Whitehead in his paper on *Mathematical Concepts of a Material World*; and the modern theory of Relativity in Electrodynamics suggests that our old plan is not the best for dealing with all the facts.[1]

Let us now return to our analogy between space and the parameter θ used in dealing with the cycloid. What is meant by asking: Is physical space real and is it Euclidean? If a man asked whether the parameter θ really exists he would mean: Did this particular physical cycloid originate through the rolling along a physical straight line of a physical circle provided with some arrangement for making a mark? In another sense the parameter θ is always real in the case of any real cycloid; for it is a definite function of magnitudes connected with this cycloid which can actually be measured. In no sense is the parameter θ subjective; it is not made by us arbitrarily; this function of the Cartesian co-ordinates of a cycloid exists whether we notice it or not, and the only subjective part of the whole business is our determination for purposes of our own to denote this function by a definite letter and to deal with it explicitly. Let us apply this analogy to space. What is meant by asking whether space is real? As with the question about the parameter θ the question is ambiguous. It may mean (1) Are the points of physical space of the same logical type as particular sense-data? Sense-data are particulars, not classes or relations. The question thus means: Are the points of physical space particulars like sense-data or are they classes or classes of classes or relations? The second meaning of the question is: Can all observable movements be stated as functions of physical bodies with the qualities that have been ascribed to them and of space with the qualities that have been ascribed to it in the particular system of physics and geometry under discussion? If so the space and the

[1] *Cf.* Mr. A. Robb's very interesting and important work, *A Theory of Time and Space.*

physical bodies of this system of physics and geometry are real in the sense that they are definite functions of observable facts, functions which of course are just as real whether we explicitly notice them and give them names or not.

Now can these questions be answered? The second can be answered by simple inspection. You have merely to see whether there be any facts of sensible experience that will not fit into the system of physics and geometry under discussion. If they all do so you will be able to say that the space and the bodies of this system are real, as far as we have been able to tell, in the second sense. I can hardly give a better example of what I mean than by asking whether the facts, of electrodynamics fit in with the Newtonian physics and the Euclidean geometry. If they do not then either Euclidean space or Newtonian bodies are not real in the present sense. Now, in so far as we have in electrodynamics to assume the Lorentz-Fitzgerald contraction and cannot explain it physically, we shall have proved that the Newtonian physics and the Euclidean geometry are not capable consistently of dealing with all observable facts, and so that either Euclidean space or Newtonian bodies are not real. For if the Lorentz-Fitzgerald contraction cannot be explained physically it will involve an action of space or matter, and this is contrary to the general conditions which all space must obey. I do not say that the facts of electrodynamics do force us to conclude that either Euclidean space or Newtonian bodies are unreal in the present sense; but I take this as an illustration of the sense of reality under discussion, and remark in passing that these facts have actually led certain mathematicians and philosophers, *e.g.* Minkowski and Mr. Robb of Cambridge, to elaborate a new system of geometry and a new system of physics which shall consistently fit all the facts.

Let us now return to the first meaning of reality. The first thing to notice is that the question whether space and bodies be real merely in the sense of being of the same logical type as objects of perception is of very little importance compared with the question whether they be real in the sense of being consistent functions of all observable facts. And I think the question is and will always remain an unanswerable one. The only way in which we can become sure of the existence of any particular, *i.e.* of anything of the same logical type as a sense-datum is either (1) by actually perceiving it or (2) by inferring its existence as causally connected with some of our sense-data. We have seen that space cannot be known in either of these ways. It might perhaps be said that at any rate physical bodies can be inferred as the causes of certain sense-data, *e.g.* light waves are inferred as the causes of our sensations of colour. I am now inclined to think that this is possibly a mistake and that Mr. Russell may perhaps be right in regarding physical bodies as constructions just as space is a construction. The only difference will be that space is a construction involving objects of a higher logical type; thus if physical bodies be classes of sense-data, points of space will be classes of classes of sense-data. So the only answer to the question whether points of space be real in the same sense as perceived objects is the following: Geometrical points and physical bodies as constructed by us are certain functions of our sense-data. Geometrical relations of spatial points are relations between these functions so determined as to be consistent with the relations that we actually can observe between sense-data themselves. Whether there be particular entities as distinct from these functions which have to each other the same relation as these functions we cannot tell. If there be such entities then geometrical points exist in the same sense as directly experienced objects; if not they only exist in the sense in which a real function of certain existent magnitudes can be said to exist. And the latter mode of existence is enough for all scientific purposes, and enough to make the existence of space independent of any one's mind.

We are now in a position to answer the question which forms the title of this paper: What is meant by asking whether our space is Euclidean? All that it can really mean is this: Subject to the conditions that space is to be changeless and homogeneous and not to act on matter, and that matter is to move about in space, can we construct a system of physics which assumes Euclidean geometry for space and enables us to deal consistently and adequately with all the data and all the changes in the data of the various senses of all sane waking men? This way of putting the question asks a little more than we really ask of science; we actually allow science to neglect a good many sense-data, *e.g.*, those obtained through using the tip of the tongue, etc., and we also allow it to prefer some sense-data to others of the same sense; *e.g.*, we are contented if its amount is consistent with the sense-data reached by means of an instrument of precision like a microscope, and allow it to neglect, in comparison with these, sense-data obtained by the naked eye. So the final form of the question will be this: Subject to the conditions that space is to be changeless and homogeneous and not to act on matter, and that matter is to move about in space, can we construct a system of physics which assumes Euclidean geometry for space, and enables us to deal consistently and adequately with all the data that scientists agree to be most worthy to be taken into account? We cannot now go further into the long and complicated story of the justification of scientists in preferring some sense-data to others; we must take it for granted here that they are right.

Now of course the only way to answer such a question as this is actually to try and construct such a system of physics. If you can do it, well and good; space is Euclidean. If not then space may not be Euclidean. But there are three very important points to notice here. 1. It is notoriously very difficult to prove a negative. Anybody's or everybody's failure to construct a satisfactory system of physics assuming non-Euclidean geometry could not prove that such a system was impossible and that our space was not Euclidean. At the most it would strongly suggest it. 2. All the alleged particular crucial experiments like measurements of stellar triangles, of parallax, etc., are quite wide of the mark. They forget that *both* physics and geometry are constructed out of a *common* matrix, *viz.*, people's sense-data and the relations and changes of these, and therefore our physics and our geometry are essentially correlatives. Hence such experiments at best only answer the question: Can we keep our physics unchanged and retain our Euclidean geometry. This may be an interesting question, but it is not the question whether our space is Euclidean. And it is essentially an unfair kind of question. You might just as well ask: Can we keep our Euclidean geometry unchanged and retain our old physics? The proper question is: Can we make up *any* system of physics which will account consistently for all the facts and allow us to retain Euclidean geometry? if so our space is Euclidean. 3. The third very important point to notice is that, if it be decided that our space is Euclidean, this will in no way prove that it is not also non-Euclidean. If we can also make up a system of physics which shall deal consistently with all the sense-data recognised by science and assume hyperbolic geometry, then our space will also be hyperbolic. The moment that we see that physics and geometry are essentially correlative factors in a certain way of treating a common experience we see that there need be no essential incompatability between the three geometries.

Before closing this paper I would like to say a word about an argument that is used by Prof. Aliotta.[1] He seems to

[1] *The Idealistic Reaction against Science.*

hold that there is no incompatability between Euclid and the other two geometries because hyperbolic and elliptic geometry are mere fragments of Euclidean geometry. All figures dealt with by non-Euclidean geometry exist in Euclidean geometry, but some Euclidean figures do not exist in non-Euclidean geometry, *e.g.* Euclidean straight lines. As stated this is undoubtedly false. The geometry of the horosphere in hyperbolic geometry is Euclidean, just as much as the geometry of the pseudosphere in Euclidean geometry is hyperbolic. Again in hyperbolic geometry there are equidistance curves to correspond to Euclidean parallels, though of course equidistance curves are not hyperbolic straight lines. But although Prof. Aliotta's explicit statement is certainly mistaken it misses the following interesting point. In Euclidean geometry points, straight lines, and planes are simply defined by postulates, *i.e.* by propositions asserting their relations to each other. It follows that a point, a straight line, and a plane in one space cannot mean precisely the same things as points, straight lines, and planes in any other space. This at least leads to the suggestion that, *e.g.* hyperbolic points may be certain logical functions of Euclidean points and Euclidean points certain logical functions of hyperbolic points. If this be so then, even keeping our physics constant we can treat the corresponding space as either hyperbolic, elliptic, or Euclidean; and the three geometries will only be three different ways of meeting the same geometrical relation. But whether this last suggestion be a fact is a question into which we cannot now enter.