*Joint Meeting of the Cambridge University Moral Science Clubs and the Aristotelian Society on Thursday, February 19th, 1942, at 8.30 p.m., in Prof. C. D. Broad's rooms, at Trinity College, Cambridge.*

# I.—KANT'S THEORY OF MATHEMATICAL AND PHILOSOPHICAL REASONING.

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The most complete account which Kant gives of his view of the nature of mathematical reasoning is in the first section of that part of the Critique of Pure Reason which is called *The Discipline of Pure Reason.* It is to be feared that most of us, unless we are professional students of Kant, have rather flagged before reaching p. 570 of Kemp Smith's translation and are inclined to give ourselves a holiday on the plea that the rest of the book is just " Kant's architectonic." This is a pity, because it contains much of interest and importance. For this reason I am going to discuss it in the present paper.

Kant is here concerned to show the difference between the methods of mathematics and philosophy and to prove that, from the nature of the two subjects, the method of one cannot be used by the other. The argument is rather long-winded, but it seems to amount to the following: It is characteristic of mathematics to start from definitions which are obviously adequate and from axioms which are self-evident and to reach conclusions which are rendered intuitively certain by demonstration. Kant tries to prove that there can be nothing strictly analogous to this in philosophy.

(1) There is nothing in philosophy which is strictly analogous to a mathematical definition, e.g., to the definition of a circle. In the first place, no purely *empirical* concept of what Mill would call a " Natural Kind," such as the concept of gold or of water, can strictly be defined. We can indeed, if we like, take a certain set of perceived qualities as an accepted description of gold or of water. But in doing so we assume that they are reliable signs of the presence of all the innumerable other characteristic properties of these substances. E.g., the " definition " of man as an animal with two legs and no feathers is useful only in so far as we

have reason to expect from past experience that anything that has these two properties will have also all the other properties of what we commonly call a man. We have no guarantee that this will be so. If in any case this quite contingent conjunction of attributes should break down, we should hesitate to call a creature a man even if it had two legs and no feathers. Suppose, e.g., that it had two legs and no feathers and a wholly non-human shape and no power of speech or reasoning.

Secondly, the *pure concepts of the Understanding*, such as cause and substance, cannot strictly be defined. Or, rather, there can be no guarantee that any proposed definition is adequate. For these concepts are given to us in a confused form in perceptual experience, e.g., when we perceive moving stones breaking windows, fires boiling kettles, and so on ; and any attempted definition represents nothing more than the degree of analysis of such typical situations which has been reached by a given person at a given date. Such definitions are always liable to be inadequate and they may be positively erroneous.

According to Kant there is one and only one condition under which we can be certain that our definitions are both adequate and correct. This is when we arbitrarily make up the concept for ourselves. In such a case, he says, I must know what I have intended to think about in using the concept. But the question at once arises : What is the use of making up arbitrary concepts and defining them when they may apply to nothing ? To this he answers that there is one and only one case where this difficulty can be avoided. " There remain, therefore, no concepts which allow of definition except only those which contain an arbitrary synthesis that admits of *a priori* construction."

What Kant means is this. (i) Suppose I define a circle as a line all of whose points are equidistant from a fixed point. Then I am not attempting to analyse a concept which I have derived by abstraction from perceived instances of it nor one which is given to me *a priori* in a confused form. All that I am saying is : " I propose to think about a line all of whose points are equidistant from a fixed point, and for brevity I shall call such a line a ' circle '." (ii) I can actually construct a figure answering to this definition and see it with my mind's eye even if I have never seen a physical object answering to this definition. For, according to Kant, each of us carries about with him a mind-dependent *intuitum* in which he can imaginatively construct without help from sensible experience any figure that he may choose to define. (iii) Kant seems to me seldom to draw a sharp or clear distinction between pure and applied mathematics. But I think that his argument might be continued as follows: For pure mathematics all that is necessary is this imaginative construction of something intuitable which answers to the concept. But this construction takes place in the innate spatial framework which each person carries about with him as a perpetually intuited mind-dependent object. Now particular empirical objects get their spatial characteristics simply through the arrangements of actual sense-impressions (as distinct from self-initiated image-elements) in the very same innate spatial framework. So each of us can be sure that anything which is true of the pure circles, which he constructs imaginatively apart from sensation, will be true of any circular empirical object that he may ever perceive with his senses.

(2) In philosophy there are no self-evident synthetic propositions. Therefore there are no axioms in the mathematical sense. Kant's view is that a necessary connexion between two concepts *a* and *b* can never be discerned by merely reflecting on them and comparing them, except in the trivial case where *a* is of the form *bx* or *b* is of the form *ay*, where *bx* and *ay* are mere conjunctions of concepts, such as red-and-round.

Now with mathematical concepts we can construct in imagination an instance of the two concepts, and on inspecting this we may be able to see directly a necessary connexion between the concepts. According to Kant, e.g., no amount of reflexion on the two concepts of straightness and distance would enable one to see that the straight line joining two points is the shortest distance between them. But you have only to imagine two points and then to imagine a straight line and other lines joining them, and you will see on inspection that the two properties of straightness and shortest distance are necessarily connected.

But the concepts of philosophy, e.g., cause, substance, etc., cannot be instantiated intuitively in imagination ; and therefore, according to Kant, no synthetic philosophical proposition, such as " every event has a cause," can be seen to be necessary by reflexion or inspection. All such propositions need proof, and their proof always consists in showing that, unless they were true of empirical objects, no such experience as we have could arise from the kind of data with which we are presented. The essential mediating term in all such cases is the fact that human experience exists and has certain very general kinds of systematic order in spite of its data being scrappy, subjective, and initially disconnected sense-impressions. This fact, as Kant recognises and asserts, is logically and metaphysically contingent ; and therefore anything that depends on it is contingent in these senses, though it may be called " transcendentally necessary."

(3) Outside mathematics there are no genuine demonstrations, for the same reason that there are no genuine axioms elsewhere. Kant is here using the word " demonstration " in a technical and narrow sense. We commonly use it in philosophy as equivalent to any kind of conclusive proof. But Kant means by a " demonstration " quite literally " a pointing out so that it can be seen by the mind's eye." His view is that in a mathematical proof each stage simply consists in putting the reader in a position to see directly with his mind's eye the truth and necessity of some proposition which he was not in a position to see before.

He takes as an example the ordinary proof in Euclid that the sum of the angles of a plane triangle is equal to two right-angles. Take, e.g., the triangle ABC. The first step is to produce a side, e.g., BC to D. This enables the reader to see at a glance that the angles BCA and ACD are together equal to two right angles. The next step is to draw a line CE, within the external angle ACD, parallel to the side AB which faces the angle ACB. The reader can then see at a glance the following facts : (i) that the angle ACD is equal to the sum of the angles ACE and ECD, (ii) that the angle ACE is equal to the alternate angle BAC, and (iii) that the exterior angle ECD is equal to the interior opposite angle ABC. When once he has grasped all these facts all is over except the shouting, and the rest of the proof is merely analytical.

This is a very happy example for Kant's purpose, and we must agree with him that nothing in the least like this procedure is possible in philosophy where we are concerned with concepts which are not able to be instantiated in intuition or imagination.

I will now make some comments on Kant's theory of mathematics. It was evidently made up primarily to deal with geometry, and was then extended forcibly to deal with arithmetic and algebra. Kant was struck by the part played by construction in Euclidean geometry and came to the conclusion that in all mathematical reasoning there must be constructions of some kind. I will now consider the theory in turn for geometry, arithmetic and algebra.

*Geometry.*—(1) Kant is no doubt right when he says that the properties of a triangle do not follow logically from the mere defining property of being a plane figure bounded by three straight lines set end to end so that the end of the third coincides with the beginning of the first. The figure thus defined has to be thought of, not as an independent isolated set of points, but as a certain set marked out for special consideration from the whole collection which compose the space in which it lies. This becomes peculiarly obvious whenever we make a geometrical construction. For the moment we begin to draw or to imagine lines which are not included in the definition of the original figure we are presupposing more than is contained in that definition, viz., the space in which the figure lies and in which these auxiliary lines exist.

We can reinforce this argument in a way which was not open to Kant. The definition of a triangle is the same for every system of geometry. But many of the properties of a triangle are different according to the nature of the space in which it is supposed to be, e.g., the sum of the angles of a plane triangle is equal to two right-angles in Euclidean geometry, less than two right-angles in hyperbolic geometry, and greater than two right-angles in elliptic geometry. If we look back at the construction by which Euclid and Kant profess to demonstrate that the sum of the angles of a triangle is equal to two right-angles, we shall see that it presupposes certain things about the space in which the triangle lies. We are told to draw a line CE parallel to AB, and it is tacitly assumed that there is one and only one such line. This is true if the space is Euclidean, for this is part of the definition of a Euclidean space. But, if the space be elliptic, the construction is impossible ; for there are no parallel lines in such a space. (This can be illustrated if you try to make a similar construction with spherical triangle instead of a plane triangle in Euclidean space. The analogy to straight lines here is great circles, and you cannot draw a great circle parallel to another great circle on a sphere.) Again, if the space be hyperbolic, the construction, though possible, is useless. For in such a space the angle ACE is *not* equal to the alternate angle BAC and the external angle ECD is *not* equal to the interior opposite angle ABC.

It is evident then that the properties of a figure do not follow from the concept of that figure alone. But, although they do not, Kant is not justified in concluding that something beside concepts is needed. The properties of a figure *do* follow deductively from the concept of the figure together with the concept of the space in which the figure is supposed to be. And the latter concept just consists of the set of axioms or postulates which together express the defining properties of this space. It is true that many of Euclid's propositions do not follow deductively even from his definitions, axioms, and postulates, taken together, and that intuition is needed in some of his proofs. But that is only because Euclid did not introduce enough explicit axioms and postulates, i.e., did not carry his conceptual analysis of Euclidean space far enough. Kant seems to have mistaken a defect in the geometry of his time for an inherent property of geometry as such.

(2) Suppose that Kant had admitted that everything in geometry can be deduced without further appeal to intuition provided that we start with enough explicit axioms and postulates. He might then have fallen back on a

second line of defence. He might have argued that intuition is needed in order to guarantee the axioms and postulates. In order to deal with this suggestion we must draw a distinction which was not recognised in Kant's time.

There was at that time one and only one system of geometry known, viz., Euclid's ; and it was assumed that this is " the geometry of nature." The distinction between different kinds of pure geometry had not been made, and therefore the distinction between pure geometry and the special geometry of nature was not recognised. Geometry was, in this respect, in the position in which arithmetic is now. We know, however, that there are a number of different systems of pure geometry, each of which is internally consistent. Until we begin to apply them to nature they are all on a level. The propositions by which we define the various types of space, e.g., Euclidean, hyperbolic, and elliptic, are taken, not as axioms, but merely as postulates, i.e., as propositions which we suppose in order to develop their logical consequences. And the propositions which we prove are all hypothetical. They are not of the form " The sum of the angles of a plane triangle *is* equal to two right angles." They are of the form " The postulates which define a Euclidean space entail that the sum of the angles of any plane triangle in such a space are together equal to two right-angles."

It is clear that so long as we remain at the level of pure geometry there is no need of intuition to guarantee our axioms. For we are not asserting these propositions ; we are simply supposing them in order to draw consequences from them. What is merely supposed and not asserted does not need a guarantee of its truth.

There are just two small qualifications to be made to the above statement. The first is merely psychological, the second is logical. (i) Of course mathematicians did not at first assume sets of postulates merely at random. The historical fact is that they started from Euclid's axioms and postulates, which were accepted as an accurate description of the space of nature. Then difficulties arose about the axiom of parallels, and it was found that a consistent system of geometry could be produced by denying this axiom and keeping the rest intact. Gradually geometers became bolder, and now they work out the consequences of sets of postulates which define types of space extremely unlike anything that we can perceive or image. So it is true that intuition supplied the first stimulus to geometry, and that geometry has only very gradually cut itself loose from the spatial characteristics of perceptible and imaginable objects. Still, the separation is now fairly complete, and geometry may now be regarded as the science of continuous aggregates of more than one dimension.

(ii) The second, and logical, qualification is the following. A set of postulates cannot be asserted absolutely at random if any important result is to come from them. They must at least be consistent with each other. So it is always necessary to establish the self-consistency of a set of postulates before beginning to deduce anything from them. Now one way of doing this is to point to something which we can actually perceive and can directly see to fulfil all the postulates of the set. Since this something exists and fulfils all the postulates, there can be no inconsistency between them. In such cases we may say that intuition guarantees, not the truth of the several postulates, but their consistency with each other. In general, however, mathematicians are not able to guarantee the consistency of a set of postulates by pointing to a perceptible thing, or a group of such, which visibly obeys them all. Their usual procedure has been as follows. They take it for granted that there is no contradiction in the system of real numbers. They then try to define a class of real numbers which shall answer to a set of postulates logically equivalent to those of the system of geometry which they are testing for consistency. If they can define such a class of real numbers, they conclude that their set of geometrical postulates is self-consistent.

Now, in a very wide sense, this process might be called " constructing " (or, better, " instantiating ") " the concepts of geometry in the number-system." And in this very wide sense we may admit that even pure geometry needs to " construct its concepts." But we must remember (*a*) that to instantiate them in terms of certain classes of real numbers and their arithmetical relations can hardly be called constructing them in *intuition* unless we accept Kant's view of the nature of arithmetic, which we have not yet discussed ; and (*b*) that the construction is needed only in order to prove the *consistency* of the postulates with each other and not to prove the *truth* of any one of them.

There is therefore extremely little to be said for Kant's theory as applied to pure geometry. If his ghost were to accept the above remarks, after studying the later developments of the subject, it might now retire to a third line of defence. He might say : " I grant that geometrical *reasoning* needs no appeal to intuition, if we start with enough explicit postulates. I grant too that, so long as we are content to consider the consequences of sets of postulates which are supposed and not asserted, there is no need of intuition to guarantee the *truth* of the several postulates, though it may be useful in order to guarantee their *mutual consistency*. But in the case of one system of geometry, viz., Euclid's, we go further than this. We are not content to say that Euclid's propositions are necessary consequences of the postulates which define a Euclidean space. We know that Euclid's propositions are themselves true, because we know that the postulates which define a Euclidean space are true. These postulates are not mere suppositions ; they are facts which we know about external nature. Such facts can be known only by intuiting actual instances of them either in imagination or in perception."

What are we to say of this contention ? If it were true, as Kant no doubt believed it to be, the situation would be as extraordinary as he evidently believed it to be. It would mean that we are able to know with complete certainty, and not just probably by means of induction, the fundamental facts about the spatial structure of nature at all places and throughout all time. As we know, Kant thought that there is one and only one view of space, viz., that the percipient imposes spatial characteristics on the objects which he perceives, which would explain the possibility of such knowledge. It seems to me quite certain that, even if this extremely paradoxical view of space were acceptable on other grounds, it would not account for the kind of knowledge which we are alleged to have of the spatial characteristics of nature. I do not propose to go into that question here ; instead I shall raise the question whether we really have such knowledge at all. Are the axioms of Euclid synthetic propositions about the spatial structure of external nature ? And do we know these propositions with complete certainty through constructing instances of them in an innate spatial *intuitum*, or are they just empirical propositions which are rendered very probable by induction ?

It seems to me that we may distinguish three classes among the axioms of Euclid. (i) Those which express peculiar features of Euclidean space and distinguish it from other kinds of homaloidal space. An example is the postulate that through a point outside a straight line one and only one straight line can be drawn which is co-planar with the former and does not intersect it in either direction. (ii) Those which express features common and peculiar to all forms of homaloidal space, whether Euclidean, elliptic, or hyperbolic. An example would be what Russell calls the " axiom of free mobility," i.e., roughly that a figure of given size and shape which can exist in one region of the space can exist equally in any other region of it. (iii) Axioms which are common to every kind of geometry, including the geometries of non-homaloidal spaces. My opinion is that axioms of the first kind are partly empirical and partly conventional ; those of the second kind are analytic in a rather special sense, which I will explain ; and those of the third kind occupy a rather peculiar position about which I do not feel altogether clear. I will now take them in turn.

(1) The axiom of parallels can hardly be called self-evident, for it was just because people found themselves able to doubt it that they were led to develop systems of non-Euclidean geometry. Nor can it be said that intuition provides any direct evidence on the subject. We can neither perceive nor imagine straight lines of indefinite length, whilst the question of parallels is a question about co-planar straight lines which would never intersect no matter how far they were produced.

This, however, is not perhaps conclusive. There is here a peculiar situation which is of considerable logical and epistemological interest, and it is one that I have never seen discussed. It may be put as follows: Let us call the axiom of parallels P and the other axioms A. Then there are numerous other propositions, of which we will take Q to be typical, having the following properties: (*a*) PA together entail Q. (*b*) QA together entail P. (*c*) Q would appear to many people to be self-evident. An example of Q is the proposition that there can be figures which are similar in shape and different in size. This is entailed by the axiom of parallels together with the other axioms, and it, together with the other axioms, entails the axiom of parallels. I think that many people who would not find the axiom of parallels self-evident would find it self-evident that there can be figures which are similar in shape and different in size. What is the right course of action in such a case? Ought we to doubt the necessity of the proposition about similar figures, in spite of its seeming evident when viewed by itself, because it in conjunction with the other axioms entails something that we can quite well doubt? Or ought we to accept the necessity of the axiom of parallels, in spite of the fact that it does not seem evident when viewed by itself, because it is entailed by the conjunction of the other axioms with something which we find self-evident on inspection? I do not think that any difficulty arises unless the other axioms are found to be self-evident. But many people would claim to find all the axioms of Euclid except that of parallels self-evident; and for such people the question which I have just raised is important. The first of the two alternatives which I mentioned is the safer course if we want to avoid believing propositions which may be false; the second is the safer if we want to avoid disbelieving propositions which may be true.

If we do not find any propositions such as Q self-evident, or if we prefer to follow the more ascetic of the two alternatives mentioned above, we shall have to say that the evidence for the axiom of parallels is empirical, but indirect and not intuitive. The position is as follows: The laws of geometry and the laws of physics together form a single hypothesis put forward to correlate the observable facts of sight, touch, movement, etc. If you take the present laws of physics as fixed, there is empirical evidence for Euclidean geometry. But, if you are prepared to alter the present laws of physics in certain respects, you can still correlate all the observable facts by making suitable modifications in the axioms of geometry. The situation is analogous to a single equation with two variables, e.g., $ax + by = c$, where $c$ represents the observable facts, $x$ the axioms of geometry, and $y$ the laws of physics. If $x$ be taken as fixed, $c$ determines $y$; and if $y$ be taken as fixed, $c$ determines $x$. But, if you allow yourself to vary both $x$ and $y$, $c$ determines only the way in which any change in the one must be compensated for by a change in the other. This is what I mean when I say that the evidence for the peculiar features of Euclidean geometry, such as the axiom of parallels, is partly empirical and partly conventional.

(2) The axiom of free mobility seems to me to be involved in the distinction between space and things which occupy space. For it really amounts to saying that *mere* difference in position makes no difference to the shape or size of a body. Now this denial of causal action is the only way in which space could be distinguished from a material medium distributed throughout space. The axioms which are common to all forms of homaloidal space and peculiar to them are therefore analytic in the sense that they are part of what we mean by talking about space as distinct from materials which occupy space.

Now some philosophers have held that knowledge of general facts about nature would be impossible unless this distinction could be drawn. This seems to have been held by Russell in his *Foundations of Geometry* and by Whitehead in his *Principle of Relativity*. On the other hand, it is implicitly denied by most of the exponents of the General Theory of Relativity, though many of them write so carelessly and metaphorically that it is extremely difficult to be sure what they really mean. If Whitehead could make out his case, we could say that these axioms are transcendentally *a priori* though logically and metaphysically contingent. For we could argue that we do know certain general facts about nature; that this would be impossible unless space were homaloidal; and that these axioms just state in abstract terms what is involved in the notion of a homaloidal space. I regret to say that I cannot follow Whitehead's arguments, and I am not convinced by Russell's earlier arguments. It seems to me doubtful that this way of treating natural phenomena is an essential condition of the possibility of knowledge of general facts about nature. Therefore I am not prepared to admit more than that these axioms are necessary consequences of a certain way of treating natural phenomena scientifically which had proved highly successful up to about 1925. So I should not call these axioms even transcendentally *a priori*.

This raises a rather curious point in connexion with Kant's theory. It is plain that he puts the axioms of geometry, which are intuitively certain, on a higher level than the Principles of Pure Understanding, such as the law of universal causation, which need a transcendental proof. The reason is that the latter have to appeal to the purely contingent fact that a coherent experience exists. He seems to ignore the fact that, on his view, the axioms of geometry depend on something still more contingent, viz., that our minds are provided with that particular form of spatial intuition which they happen to have. I call this *more* contingent because it is perfectly obvious, and is admitted by Kant himself, that an orderly experience could exist if we had a different form of spatial intuition, and perhaps (though this is more doubtful) even if we did not impose spatial characteristics on our sense-impressions at all. Yet, if Kant's alleged proofs of the Analogies of Experience were valid, nothing worth calling "experience" would be possible unless our sense-impressions could all be synthesised in such a way that every empirical event can be regarded as fully determined by earlier empirical events. Thus the reasons which he gives for putting the axioms of geometry above the Principles of Pure Understanding should really put the latter above the former. It is a defect and not a merit of the axioms of geometry not to have a transcendental proof. If Whitehead were right, the axioms which assert that space is homaloidal would avoid this objection, for they would be capable of a transcendental proof.

(3) The axiom that the shortest distance between two points is the straight line which joins them is common to all systems of geometry, whether homaloidal or not.

The first thing to notice about it is the following: The notion of the shortest path between *p* and *q* is not determinate until we specify the conditions that are to be fulfilled by the intermediate points on the various alternative paths. This fact can easily be illustrated as follows: Suppose that the points *p* and *q* are on the surface of a sphere. Then the shortest path between them, whose points are wholly confined to the surface of that sphere, is the great circle which passes through *p* and *q*. But *p*, *q*, and all the intermediate points on this great circle, also belong to the three-dimensional Euclidean space in which this sphere is a two-dimensional surface. Now the great circle is not the shortest distance between *p* and *q* if we are allowed to take account of paths which are not confined to the surface of the sphere though they are contained in the three-dimensional Euclidean space in which the sphere is a surface. The shortest distance, under these less restricted conditions, is the Euclidean straight line joining *p* and *q*. Similarly a three-dimensional non-Euclidean manifold of points can be regarded as a certain selection from a four-dimensional Euclidean manifold of points. If *p* and *q* are two points in the former, they will also be two points in the latter. The shortest of all the paths between *p* and *q* which lie within the former manifold will not necessarily be the shortest of all the paths between them which lie within the latter. It is evident then, that the axiom must be restated as follows: "The shortest of all the paths from *p* to *q* which are confined to the space S is the straight line joining *p* to *q* in the geometry of that space."

Now in the case of non-homaloidal spaces I suspect that this axiom is merely a definition of "straight line" for the space in question. I should doubt whether any meaning can be attached to the notion of straight line in the geometry of such spaces except that of geodesic or shortest path. The qualitative aspect of the concept of straightness would seem to have vanished here. I may be mistaken about this, and competent geometers will no doubt correct me if I am.

In the case of homaloidal spaces I do not think that the axiom is a mere definition. In such spaces the straight line has certain purely qualitative characteristics. It is, e.g., the one and only kind of curve in the space which has the following properties. (i) Each such curve is completely determined by a pair of points. (ii) Every pair of points determines one such curve. (iii) Any one such curve is equally determined by any pair of points on it. Thus, in a homaloidal space at any rate, the notion of straight line has, as Kant alleged, a purely qualitative meaning ; and indeed in projective geometry no quantitative notions are attached to it.

It does not follow that the axiom that the shortest distance between two points in a homaloidal space is the straight line in that space which joins them is a synthetic *a priori* proposition. It is evident that the proposition implies the possibility of comparing the lengths of various non-straight paths with each other and with the length of a straight path. This in turn implies some definition or convention for the measurement of the length of a non-straight line. Now I think that this will be found to involve the notion of replacing the non-straight line by a chain of short straight lines connecting successive closely adjacent points on it, summing the lengths of these, and then proceeding to the limit as the length of each is reduced and the number of all is increased indefinitely. If so, length as a measurable quantity applies primarily *only* to straight lines ; when it is applied to non-straight lines it has to be defined in terms of length as applied to straight lines. Assuming that this is correct, it would seem that the axiom comes down to repeated applications of the proposition that any two sides of a triangle are together greater than the third side. What are we to say about this proposition ? It requires some definition or convention for comparing the lengths of straight lines which differ in direction. I do not know precisely what this convention is ; but I am very much inclined to think that, if it were made explicit, we should find that the proposition that two sides of a triangle are together greater than the third side is either an analytic consequence of it or a synthetic empirical proposition about the results of certain physical processes performable with measuring-rods or light-waves in the actual physical world.

This concludes what I have to say about the nature of geometrical definitions, axioms, and reasoning in reference to Kant's theory about them. I pass now to his views about arithmetic.

*Arithmetic.*—The case of arithmetic is much less complicated than that of geometry for the present purpose. In the first place, there is no question of a distinction between several alternative internally consistent and mutually incompatible systems of arithmetic. Therefore there is no obvious need to raise a separate question about pure arithmetic and the arithmetic of nature. Now the paradox of geometry was that geometrical propositions seemed to be *a priori* and yet to give information about the spatial structure of the actual world. No such paradox arises in connexion with arithmetic. The propositions of arithmetic would presumably apply in all possible worlds, and so there is no reason to doubt their *prima facie* claim to be *a priori.* The only question that we need raise is whether they are synthetic.

Kant firmly maintained that they are. He says that, however much you may reflect on the definitions of the numbers 7 and 5 and 12 and on the definition of addition, you will never be able to see that $7 + 5 = 12$. In order to do so you must instantiate the concepts by means of dots, real or imaginary, and then you will see, not only that this set of 7 dots and this set of 5 dots make a set of 12 dots, but also that the same will necessarily be true in all such cases. We may make the following comments.

(1) Kant never tells us what he supposes to be the definitions of 5 and 7 and 12 and of addition, or whether he believes them to be indefinable. He merely makes the negative remark that no amount of reflexion on the concepts themselves would enable one to see that $7 + 5 = 12$. This is all the stranger when we remember that Leibniz had professed to prove such propositions and that Kant was thoroughly versed in Leibniz's philosophy. Leibniz starts by defining " 2 " as $1 + 1$, " 3 " as $2 + 1$, " 4 " as $3 + 1$, and so on. Let us, for shortness, consider how he would

claim to prove that $3 + 2 = 5$. If this were valid, a proof that $7 + 5 = 12$ could easily be constructed on the same lines ; it would merely be rather longer. The argument would run as follows :—

$$5 = 4 + 1 \text{ Def. (i)}$$
$$4 = 3 + 1 \text{ Def. (ii)}$$
$$2 = 1 + 1 \text{ Def. (iii)}$$

Combining defs. (i) and (ii) we have

$$\begin{aligned} 5 &= (3 + 1) + 1 \\ &= 3 + (1 + 1) \\ &= 3 + 2 \text{ by def. (iii).} \end{aligned}$$

Is there anything wrong with this proof, and, if so, what? As Kant does not mention it or criticise it, we must consider it for ourselves. The following comments may be made on it. (*a*) The sign " = " has different meanings in the course of the argument. In the definition $5 = 4 + 1$, if it really is a definition in the sense in which Kant would use that word, " = " must be a symbol for the word " means," and the sentence should read : " ' 5 ' means the sum of 4 and 1." Now it certainly cannot have this meaning in the intermediate steps or in the conclusion. If " 5 " means the sum of 4 and 1, then it plainly does not *mean* the sum of 3 and 2. Therefore the sign " = " must be interpreted differently in the conclusion $5 = 3 + 2$ and in the definition $5 = 4 + 1$. Some account is required of what the sign " = " stands for in the intermediate steps and the conclusion, and some justification is needed for the transition from its use in one sense in the premiss to its use in this other sense in the conclusion. (*b*) What is the nature and justification of the step from $5 = (3 + 1) + 1$ to $5 = 3 + (1 + 1)$, which occurs in the course of the argument? It would commonly be said to be made in virtue of the Associative Law in arithmetic. But this is merely to give a name to it. Might not Kant say that we derive our evidence for this law from reflecting on operations with real or imaginary dots, and that there is no other way of getting to know it?

(2) Let us now leave Leibniz and consider the matter for ourselves. It is certain that when we do an addition

sum we do not in fact perform any operation in the least like that suggested by Leibniz. If the individual numbers and their sum is below 10 we just remember what we have learned about the sums of such numbers and write the result down without thinking. The evidence which we originally had for believing that the sum of such and such a pair of numbers below 10 is so-and-so was, as Kant says, intuition of fingers or dots or counters. When the numbers are 10 or greater than 10 we write them down in the Arabic system of notation and proceed to operate with them unthinkingly in accordance with certain rules which we were taught at our kindergartens and accepted without explanation along with the collects and the catechism. There is, of course, a reason behind these rules, and we must now state it.

The Arabic notation rests upon a certain general proposition about integers, which can be proved without much difficulty, though I do not propose to prove it here. It may be stated as follows : " Every different number which is not less than a given number $s$ can be uniquely described by a different instance of the general formula

$$a_n s^n + a_{n-1} s^{n-1} + \ldots\ldots + a_1 s + a_0,$$

where the $a$'s can be any numbers less than $s$ including 0." In the usual form of the Arabic notation $s$ is taken to be the number ten ; this leaves the numbers from nought to nine undescribed by this method, so that the first ten number-symbols are pure proper names ; but it gives a general method of uniquely describing every number above nine. If we like, we can take $s$ to be the number two, in which case the only numbers left undescribed will be nought and one. Two will then be represented by the symbol 10. Once such a general method of describing all numbers in terms of a few of the smaller ones has been established the rules for adding numbers not less than $s$ can easily be deduced, and then they can be learned by heart and followed blindly by persons who have no idea of the reasons for them. The problem of adding two numbers $x$ and $y$ comes simply to the following : " Given the description of $x$ and the description of $y$ on this system, to find the description on this system of the number which is their sum." The rules of arithmetic give the method of solving such problems.

(3) What then is the position of intuition? It is only by operating with dots or counters or fingers that we can originally discover those propositions about integers which enable us to set up a single uniform system of descriptions of all numbers not below $s$ in terms of numbers below $s$. After this notation has been set up and the rules for operating it have been deduced there is no need for any further appeal to intuition in the original sense, i.e., to what Kant calls the " construction of concepts in intuition." We no longer need to think of collections of dots or of fingers having the numbers in question. But in a wider sense we still have to appeal to intuition, viz., to the actual figures on paper or in our mind's eye by which we symbolise, on the principles of notation which we are assuming, the numbers with which we are dealing.

I am thus willing to accept a good deal of what Kant says about arithmetic if I am allowed to put my own interpretation on it. I agree that intuition of instances is necessary to make us recognise arithmetical propositions, and I agree that such propositions are nevertheless not empirical generalisations of perceived physical facts. Kant may have meant to recognise that the nature of the intuition changes when we begin to deal with larger numbers, though he certainly expresses himself very obscurely on this point. He says, e.g., in one passage in the *Transcendental Analytic*, . . . " Counting, as is easily seen in the case of the larger numbers, is a synthesis according to concepts, because it is executed according to a common ground of unity, as for instance, the decade." I admit, however, that this, like many of Kant's remarks, might mean almost anything, though it probably does not mean nothing.

(4) Beyond this my agreement ceases. I do not agree that the logically *a priori* character of arithmetical propositions is due to the fact that we can construct instances of arithmetical concepts in a psychologically *a priori*, i.e., mind-dependent, *intuitum*, viz., time or space-and-time. For, even if we did intuit such an innate *intuitum* and could

construct instances of numbers in it, this would be irrelevant to the *a priority* of arithmetical propositions. Indeed, if arithmetic depended on the peculiar structure of our sensibility, it would surely be radically contingent, whereas it seems to be intrinsically necessary. Again, since Kant's only ground for thinking that arithmetic is connected specially closely with time seems to be his view that successive moments of time provide us with something innate and mind-dependent to count and to add, I reject his reasons for connecting arithmetic with time. I also reject his conclusion. So far as I can see, there is in fact no specially close connexion between arithmetic and time.

So much for Kant's view on arithmetic. Let us now consider in conclusion what he has to say about algebra.

*Algebra.*—Kant has very little to say about this subject. I can find only two passages in the Critique of Pure Reason which bear on it. The first is in the section called *The Discipline of Pure Reason*, under the heading of *Demonstrations.* "Even the method of algebra with its equations, from which the correct answer, together with its proof, is deduced by solving them, though not indeed geometrical in nature, is yet constructive in its own characteristic way. The concepts attached to the symbols . . . . are presented in intuition ; and this method . . . . secures all inferences against error by setting each one before our eyes." The other passage about algebra occurs at the end of the sixth paragraph of the section entitled "The Discipline of Pure Reason in its Dogmatic Employment." It runs as follows. ". . . . In algebra, by means of a symbolic construction, just as in geometry by means of an ostensive construction (the geometrical construction of the objects themselves), we succeed in reaching results which discursive knowledge could never have reached by means of mere concepts."

Now it seems to me that Kant has completely changed the meaning in which he is using the term "constructing a concept in intuition," and that in the present sense it has no relation to the theory of mathematical reasoning which he has given before. Let us consider the changes that have taken place. In geometry "constructing a concept" meant actually drawing or imagining a figure answering to one's concept. In the arithmetic of small numbers it meant making a real or imaginary collection of dots or counters or fingers which has the number in question. We might call these "*instantial* constructions," meaning that the concept is symbolised by producing on paper or before the mind's eye a concrete instance of it. At this stage the theory is that geometry and arithmetic are *synthetic* because we can make instantial constructions of geometrical and arithmetical concepts ; and that they are *a priori* because we are provided with the innate mind-dependent *intuita* of space and time in which we make these constructions at will by a mere exercise of the imaging power without any need of sense-impressions from outside.

The naïve simplicity of this theory has already begun to fade when we deal with the arithmetic of large numbers. The intuited object by means of which we construct the concept of a number like 2765 is admittedly not instantial ; it is certainly not an image or a drawing of 2765 dots. It is only the perceived or imagined symbol for this number in the Arabic system of notation ; i.e., it is a concrete symbol for a particular number, constructed according to a general rule which enables us to construct symbols for every number above nine. If we adopt the Arabic system and the scale of ten, the figures from 0 to 9 are arbitrarily chosen proper names for the numbers from nought to nine ; and the figures with more than one digit are symbols which represent, according to a certain convention, definite descriptions constructed according to a general rule for each of the numbers above nine.

Now it is surely plain that Kant's original theory of mathematical knowledge, which makes such knowledge depend on the instantial construction of concepts in the innate *intuita* of space and time, does not apply at all to the non-instantial construction of concepts, which consists in writing down and manipulating symbols for these concepts according to rules. As I have said, this non-instantial construction has already begun at the stage of the arithmetic of numbers above nine. In algebra the only difference is that we begin to introduce what Johnson calls "illustrative symbols " in addition to what he calls " shorthand symbols." Our *x*'s and *y*'s stand, not for some definite number, as do the figures in arithmetic like 2 and 47, but for *any* number chosen at random as an illustration.

As a matter of fact even the figures of geometry, which we have taken as typical examples of instantial constructions of concepts, are seen, when more carefully considered, to be also of the nature of illustrative symbols. This can be seen easily as follows: There are, e.g., three kinds of triangle, viz., scalene, isoceles, and equilateral. Any triangle that we can draw or picture in our mind's eye will in fact be an instance of one and only one of these kinds of triangle. Yet, if we are reasoning about triangles in general, we shall take our figure as illustrating triangles in general and not merely that species of triangle to which it happens to belong. Again, even if we are confining our reasoning to isoceles triangles and we draw or image such a triangle, this one figure will have to represent isoceles triangles of all sizes and angles. Finally, we are quite willing to admit that our figures are seldom if ever exact instances even of the less determinate concepts which we are reasoning about. We realise that the " circles " which we draw or image are seldom if ever exact instances of the concept of circularity, and we know that this does not affect the validity of our geometrical reasoning about circles. So our figures in geometry are really illustrative symbols, like the *x*'s and *y*'s of algebra. The difference is that the latter are purely arbitrary symbols for any number taken at random, whilst geometrical figures are chosen because they at least approximately exemplify some determinate form of the generic geometrical concept which we are considering, and so serve to remind us of a number of facts about the latter and to hold them together before our minds.

Strictly speaking, the only purely instantial constructions of concepts occur in the arithmetic of small numbers when we instantiate the concept of a given number by imaging or drawing or collecting a set of objects which has that number. Here of course it is essential that the collection shall have just that number of members or it will be useless as a construction for the particular number that we want to consider.

Kant recognises these facts about geometrical figures in the numerous passages in which he insists that the intuitive construct of a geometrical concept is strictly not an image or a picture but a *schema*. Compare, e.g., the following quotation, which comes from the chapter on the Schematism of the Categories. "No image could ever be adequate to the concept of a triangle in general. It would never attain that universality . . . which renders it valid of all triangles, whether right-angled, obtuse-angled, or acute-angled . . . The schema of a triangle can exist nowhere but in thought. It is a rule of synthesis of the imagination in respect to pure figures in space."

This substitution of schema for image does not, however, remove the radical difference between the sense in which we construct concepts in geometry or in the arithmetic of small numbers and that in which we construct them in algebra or in the arithmetic of larger numbers. The schema of a triangle is not indeed an image or picture of a triangle ; but it is the general rule for *imaging or drawing triangles.* That is, it is the rule for making instantial constructions, even though it be not itself an instantial construct. But in algebra and the arithmetic of large numbers, as Kant admits, no instantial construction is possible ; and therefore there can be no schemata in the sense of rules for making instantial constructions. The schemata here can only be rules for making *symbolic* constructions ; e.g., in arithmetic the schema of the number four hundred and seventy-six would perhaps be the rule for writing down the symbol of that number in the Arabic system of notation with the scale of ten. Now Kant's original theory of mathematical reasoning presupposes that concepts are constructed *instantially* in our innate mind-dependent *intuita* of space and time. It loses all meaning when applied to the construction of non-instantial *symbols* for concepts, as distinct for accurate or at least approximately accurate *instances* of concepts.

It seems to me, then, that Kant has provided no theory whatever of algebraical reasoning or of arithmetical operations with large numbers. He has proposed a theory for geometry and for the arithmetic of small numbers. But, when we consider the differences which he admits to exist between these and algebra or the arithmetic of large numbers, we see that this theory cannot possibly be extended from the former to the latter.

Moreover, if we look at the actual remarks of Kant's about algebra which I have quoted, we see that the only explicit statements which he makes might equally well have emanated from Leibniz himself, whose theory of mathematical reasoning is utterly different from that which is typical of Kant. All that his remarks amount to is that a good system of symbolism, which makes all our assumptions explicit and can be operated mechanically according to rules, is very useful for preventing us from falling into fallacies and for enabling us to solve complicated problems. This is exactly what Leibniz said. But the question arises : Why must this be confined to the treatment of extension and number ? On Kant's earlier theory of mathematics, which makes it depend on the instantiation of concepts in the innate *intuita* of space and time, we can see why mathematical reasoning must be confined to the properties of figures and numbers. For these are the only concepts of which we can produce instantial constructions in imagination. But in his account of algebra he admits the merely symbolic and non-instantial construction of concepts. Of course these constructs themselves will be spatio-temporal, since the symbols will consist of perceived or imagined marks. But there seems no reason whatever why concepts which have nothing to do with extension or number should not be represented *symbolically* by spatio-temporal symbols. In fact everything that Kant says of algebra could be applied without change to symbolic logic ; and yet he continued to hold that there is an absolutely fundamental difference between mathematics and formal logic.